# On harmonic quasiconformal quasi-isometries

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**Abstract.** The purpose of this paper is to explore conditions which guarantee Lipschitz-continuity of harmonic maps w.r.t. quasihyperbolic metrics. For instance, we prove that harmonic quasiconformal maps are Lipschitz w.r.t. quasihyperbolic metrics.

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## 1 Introduction

Let $G \subset \mathbb{R}^2$ be a domain and let $f : G \to \mathbb{R}^2, f = (f_1, f_2)$, be a harmonic mapping. This means that $f$ is a map from $G$ into $\mathbb{R}^2$ and both $f_1$ and $f_2$ are harmonic functions, i.e. solutions of the two-dimensional Laplace equation

$$\Delta u = 0 \,. \tag{1.1}$$

The Cauchy-Riemann equations, which characterize analytic functions, no longer hold for harmonic mappings and therefore these mappings are not analytic. Intensive studies during the past two decades show that much of the classical function theory can be generalized to harmonic mappings (see the recent book of Duren [9] and the survey of Bshouty and Hengartner [7]). The purpose of this paper is to continue the study of the subclass of quasiconformal and harmonic mappings, introduced by Martio in [31] and further studied for example in [32, 33, 34, 37, 38, 16, 2, 3, 19, 20, 21, 17]. The above definition of a harmonic mapping extends in a natural way to the case of vector-valued mappings $f : G \to \mathbb{R}^n, f = (f_1, \ldots, f_n)$, defined on a domain $G \subset \mathbb{R}^n, n \geq 2$.

We first recall the classical Schwarz lemma for the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ :

**1.2. Lemma.** *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with $f(0) = 0$ . Then $|f(z)| \leq |z|$, $z \in \mathbb{D}$ .*

For the case of harmonic mappings this lemma has the following counterpart.

**1.3. Lemma.** ([15], [9, p. 77]) *Let $f : \mathbb{D} \to \mathbb{D}$ be a harmonic mapping with $f(0) = 0$ . Then $|f(z)| \leq \frac{4}{\pi} \tan^{-1}|z|$ and this inequality is sharp for each point $z \in \mathbb{D}$ .*

The classical Schwarz lemma is one of the cornerstones of geometric function theory and it also has a counterpart for quasiconformal maps ([1, 26, 41, 45]). Both for analytic functions and for quasiconformal mappings it has a form that is conformally invariant under conformal automorphisms of $\mathbb{D}$.

In the case of harmonic mappings this invariance is no longer true. In general, if $\varphi : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism and $f : \mathbb{D} \to \mathbb{D}$ is harmonic, then $\varphi \circ f$ is harmonic only in exceptional cases. Therefore one expects that harmonic mappings from the disk into a strip domain behave quite differently from harmonic mappings from the disk into a half-plane and that new phenomena will be discovered in the study of harmonic maps. For instance, it follows from Lemma 1.2 that holomorphic functions in plane do not increase hyperbolic distances. In general, planar harmonic mappings do not enjoy this property. On the other hand, we shall give here an additional hypothesis under which the situation will change, in the plane as well as in higher dimensions. It turns out that the local uniform boundedness property, which we are going to define, has an important role in our study.

For a domain $G \subset \mathbb{R}^n, n \geq 2, x, y \in G$, let

$$r_G(x,y) = \frac{|x-y|}{\min\{d(x), d(y)\}} \text{ where } d(x) = d(x, \partial G) \equiv \inf\{|z-x| : z \in \partial G\}\,.$$

If the domain $G$ is understood from the context, we write $r$ instead $r_G$. This quantity is used, for instance, in the study of quasiconformal and quasiregular mappings, cf. [45]. It is a basic fact that [43, Theorem 18.1] for $n \geq 2, K \geq 1, c_2 > 0$ there exists $c_1 \in (0,1)$ such that whenever $f : G \to fG$ is a $K$-quasiconformal mapping with $G, fG \subset \mathbb{R}^n$ then $x, y \in G$ and $r_G(x,y) \leq c_1$ imply $r_{fG}(f(x), f(y)) \leq c_2$. We call this property the local uniform boundedness of $f$ with respect to $r_G$. Note that quasiconformal mappings satisfy the local uniform boundedness property and so do quasiregular mappings under appropriate conditions; it is known that one to one mappings satisfying the local uniform boundedness property may not be quasiconformal. We also consider a weaker form of this property and say that $f : G \to fG$ with $G, fG \subset \mathbb{R}^n$ satisfies the weak uniform boundedness property on $G$ (with respect to $r_G$) if there is a constant $c > 0$ such that $r_G(x,y) \leq 1/2$ implies $r_{fG}(f(x), f(y)) \leq c$. Univalent harmonic mappings fail to satisfy the weak uniform boundedness property as a rule, see Example 2.7 below.

We show that if $f : G \to fG$ is harmonic then $f$ is Lipschitz w.r.t. quasihyperbolic metrics on $G$ and $fG$ if and only if it satisfies the weak uniform boundedness property; see Theorem 2.19. The proof is based on a higher dimensional version of the Schwarz lemma: harmonic maps satisfy the inequality (2.15) below. An inspection of the proof of Theorem 2.19 shows that the class of harmonic mappings can be replaced by $OC^1$ class defined by (3.1) (see Section 3 below) and it leads to generalizations of the result; see Theorem 3.3.

Another interesting application is Theorem 2.22 which shows that if $f$ is a harmonic $K$-quasiregular map such that the boundary of the image is a continuum containing at least two points, then it is Lipschitz. In Subsection 2.5, we study conditions under which a qc mapping is quasi isometry with respect to the corresponding quasihyperbolic metrics; see Theorems 2.25 and 2.31. In particular, using a quasiconformal analogue of Koebe's theorem, cf. [4], we give a simple proof of the following result, cf. [30, 33]: if $D$ and $D'$ are proper domains in $\mathbb{R}^2$ and $h : D \to D'$

is $K$-qc and harmonic, then it is bi-Lipschitz with respect to quasihyperbolic metrics on $D$ and $D'$.

The results in this paper may be generalized into various directions. One direction is to consider weak continuous solutions of the $p$-Laplace equation

$$div(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty,$$

so called $p$-harmonic functions. Note that 2-harmonic functions in the above sense are harmonic in the usual sense.

It seems that the case of the upper half space is of particular interest, cf. [37, 33, 16, 3]. In Subsection 2.6, using Theorem 3.1 [23] we prove that if $h$ is a quasiconformal $p$-harmonic mapping of the upper half space $\mathbb{H}^n$ onto itself and $h(\infty) = \infty$, then $h$ is quasi-isometry with respect to both the Euclidean and the Poincaré distance.

# 2 Lipschitz property of harmonic maps w.r.t. quasihyperbolic metrics

## 2.1 Hyperbolic type metrics

Let $B^n(x,r) = \{z \in \mathbb{R}^n : |z-x| < r\}$, $S^{n-1}(x,r) = \partial B^n(x,r)$ and let $\mathbb{B}^n, S^{n-1}$ stand for the unit ball and the unit sphere in $\mathbb{R}^n$, respectively. Sometimes we write $\mathbb{D}$ instead of $\mathbb{B}^2$. For a domain $G \subset \mathbb{R}^n$ let $\rho : G \to (0,\infty)$ be a continuous function. We say that $\rho$ is a weight function or a metric density if for every locally rectifiable curve $\gamma$ in $G$, the integral

$$l_\rho(\gamma) = \int_\gamma \rho(x)ds$$

exists. In this case we call $l_\rho(\gamma)$ the $\rho$-length of $\gamma$. A metric density defines a metric $d_\rho : G \times G \to (0,\infty)$ as follows. For $a, b \in G$, let

$$d_\rho(a,b) = \inf_\gamma l_\rho(\gamma)$$

where the infimum is taken over all locally rectifiable curves in $G$ joining $a$ and $b$. For a fixed $a, b \in G$ , suppose that there exists a $d_\rho$-length minimizing curve $\gamma : [0,1] \to G$ with $\gamma(0) = a, \gamma(1) = b$ such that

$$d_\rho(a,b) = l_\rho(\gamma|[0,t]) + l_\rho(\gamma|[t,1])$$

for all $t \in [0,1]$ . Then $\gamma$ is called a geodesic segment joining $a$ and $b$ . It is an easy exercise to check that $d_\rho$ satisfies the axioms of a metric. For instance, the hyperbolic (or Poincaré) metric of the unit ball $\mathbb{B}^n$ and the upper half space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ are defined in terms of the densities $\rho(x) = 2/(1-|x|^2)$ and $\rho(x) = 1/x_n$ , respectively. It is a classical fact that in both cases the length-minimizing curves, geodesics, exist and that they are circular arcs orthogonal to the boundary [6]. In both cases we have even explicit formulas for the distances:

$$\sinh\frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\,, \quad x,y \in \mathbb{B}^n\,, \tag{2.1}$$

and

$$\cosh \rho_{\mathbb{H}^n}(x,y) = 1 + \frac{|x-y|^2}{2x_n y_n}\,, \quad x,y \in \mathbb{H}^n\,. \tag{2.2}$$

Because the hyperbolic metric is invariant under conformal mappings, we may define the hyperbolic metric in any simply connected plane domain by using the Riemann mapping theorem, see for example [24]. The Schwarz lemma may now be formulated by stating that an analytic function from a simply connected domain into another simply connected domain is a contraction mapping, i.e. the hyperbolic distance between the images of two points is at most the hyperbolic distance between the points. The hyperbolic metric is often the natural metric in classical function theory. For the modern mapping theory, which also considers dimensions $n \ge 3$, we do not have a Riemann mapping theorem and therefore it is natural to look for counterparts of the hyperbolic metric. So called hyperbolic type metrics have been the subject of many recent papers. Perhaps the most important of these metrics are the quasihyperbolic metric $k_G$ and the distance ratio metric $j_G$ of a domain $G \subset \mathbb{R}^n$. They are defined as follows.

**2.3. The quasihyperbolic and distance ratio metrics.** Let $G \subset \mathbb{R}^n$ be a domain. The quasihyperbolic metric $k_G$ is a particular case of the metric $d_\rho$ when $\rho(x) = \frac{1}{d(x,\partial G)}$ (see [13, 12, 45]). It was proved in [12] that for given $x,y \in G$, there exists a geodesic segment of length $k_G(x,y)$ joining them. The distance ratio metric is defined for $x,y \in G$ by setting

$$j_G(x,y) = \log(1 + r_G(x,y)) = \log(1 + \frac{|x-y|}{\min\{d(x),d(y)\}})$$

where $r_G$ is as in the Introduction. It is clear that

$$j_G(x,y) \le r_G(x,y)\,.$$

Some applications of these metrics are reviewed in [46]. The recent PhD theses [27], [24], [29] study the quasihyperbolic geometry or use it as a tool.

**2.4. Lemma.** *([13], [45, (3.4), Lemma 3.7]) Let $G$ be a proper subdomain of $\mathbb{R}^n$.*

(a) *If $x,y \in G$ and $|y-x| \le d(x)/2$, then $k_G(x,y) \le 2j_G(x,y)$.*

(b) *For $x,y \in G$ we have $k_G(x,y) \ge j_G(x,y) \ge \log\left(1 + \frac{|y-x|}{d(x)}\right)$.*

## 2.2 Quasiconformal and quasiregular maps

**2.5. Maps of class ACL and ACL$^n$.** For each integer $k = 1,...,n$ we denote $R_k^{n-1} = \{x \in R^n : x_k = 0\}$. The orthogonal projection $P_k : \mathbb{R}^n \to \mathbb{R}_k^{n-1}$, is given by $P_k x = x - x_k\, e_k$.

Let $I = \{x \in \mathbb{R}^n : a_k \le x_k \le b_k\}$ be a closed $n$-interval. A mapping $f : I \to \mathbb{R}^m$ is said to be absolutely continuous on lines (ACL) if $f$ is continuous and if $f$ is absolutely continuous on almost every line segment in $I$, parallel to the

coordinate axes. More precisely, if $E_k$ is the set of all $x \in P_k I$ such that the function $t \mapsto u(x+te_k)$ is not absolutely continuous on $[a_k, b_k]$, then $m_{n-1}(E_k) = 0$ for all $1 \le k \le n$.

If $\Omega$ is an open set in $\mathbb{R}^n$, a mapping $f : \Omega \to \mathbb{R}^m$ is absolutely continuous if $f|I$ is ACL for every closed interval $I \subset \Omega$. If $\Omega$ and $\Omega'$ are domains in $\overline{\mathbb{R}}^n$, a homeomorphism $f : \Omega \to \Omega'$ is called ACL if $f|\Omega \setminus \{\infty, f^{-1}(\infty)\}$ is ACL.

If $f : \Omega \to \mathbb{R}^m$ is ACL, then the partial derivatives of $f$ exist a.e. in $\Omega$, and they are Borel functions. We say that $f$ is ACL$^n$ if the partials are locally integrable.

**2.6. Quasiregular mappings.** Let $G \subset \mathbb{R}^n$ be a domain. A mapping $f : G \to \mathbb{R}^n$ is said to be *quasiregular* (qr) if $f$ is ACL$^n$ and if there exists a constant $K \ge 1$ such that

$$|f'(x)|^n \le K J_f(x)\ ,\ \ |f'(x)| = \max_{|h|=1} |f'(x)h|\ ,$$

a.e. in $G$. Here $f'(x)$ denotes the formal derivative of $f$ at $x$, The smallest $K \ge 1$ for which this inequality is true is called the *outer dilatation* of $f$ and denoted by $K_O(f)$. If $f$ is quasiregular, then the smallest $K \ge 1$ for which the inequality

$$J_f(x) \le K\, l(f'(x))^n\ ,\ \ l(f'(x)) = \min_{|h|=1} |f'(x)h|\ ,$$

holds a.e. in $G$ is called the *inner dilatation* of $f$ and denoted by $K_I(f)$. The *maximal dilatation* of $f$ is the number $K(f) = \max\{ K_I(f), K_O(f) \}$. If $K(f) \le K$, then $f$ is said to be *$K$–quasiregular* ($K$–qr). If $f$ is not quasiregular, we set $K_O(f) = K_I(f) = K(f) = \infty$.

Let $\Omega_1$ and $\Omega_2$ be domains in $\mathbb{R}^n$ and fix $K \ge 1$. We say that a homeomorphism $f : \Omega_1 \to \Omega_2$ is a $K$-quasiconformal (qc) mapping if it is $K$-qr and injective. Some of the standard references for qc and qr mappings are [11], [26], [43], and [45]. These mappings generalize the classes of conformal maps and analytic functions to Euclidean spaces. The Kühnau handbook [25] contains several reviews dealing with qc maps. It should be noted that various definitions for qc maps are studied in [43]. The above definition of $K$-quasiconformality is equivalent to the definition based on moduli of curve families in [43, p. 42]. It is well-known that qr maps are differentiable a.e., satisfy condition (N) i.e. map sets of measure zero (w.r.t. Lebesgue's $n$-dimensional measure) onto sets of measure zero. The inverse mapping of a $K$-qc mapping is also $K$-qc. The composition of a $K_1$-qc and of a $K_2$-qc map is a $K_1K_2$-qc map if it is defined.

## 2.3 Examples

We first show that, as a rule, univalent harmonic mappings fail to satisfy the local uniform boundedness property.

**2.7. Example .** The univalent harmonic mapping $f : \mathbb{H}^2 \to f(\mathbb{H}^2)$, $f(z) = \arg z + i\,\mathrm{Im} z$, fails to satisfy the local uniform boundedness property with respect to $r_{\mathbb{H}^2}$.

Let $z_1 = \rho e^{i\pi/4}$, $z_2 = \rho e^{i3\pi/4}$, $w_1 = f(z_1)$ and $w_2 = f(z_2)$. Then $r_{\mathbb{H}^2}(z_1, z_2) = 2$ and $r_{f\mathbb{H}^2}(w_1, w_2) = \frac{\pi}{\sqrt{2}\rho}$ if $\rho$ is small enough and we see that $f$ does not satisfy the local uniform boundedness property.

**2.8. Example.** The univalent harmonic mapping $f : \mathbb{H}^2 \to \mathbb{H}^2$, $f(z) = \mathrm{Re}\, z \,\mathrm{Im}\, z + i\,\mathrm{Im} z$, fails to satisfy the local uniform boundedness property with respect to $r_{\mathbb{H}^2}$.

For a harmonic mapping $f(z) = h(z) + \overline{g(z)}$, we introduce the following notation

$$\lambda_f(z) = |h'(z)| - |g'(z)|, \quad \Lambda_f(z) = |h'(z)| + |g'(z)| \text{ and } \quad \nu(z) = g'(z)/f'(z).$$

The following Proposition shows that a one to one harmonic function satisfying the local uniform boundedness property need not be quasiconformal.

**2.9. Proposition .** The function $f(z) = \log(|z|^2) + 2iy$ is a univalent harmonic mapping and satisfies the local uniform boundedness property, but $f$ is not quasiconformal on $V = \{z : x > 1, 0 < y < 1\}$.

**Proof.** It is clear that $f$ is harmonic in $\Pi^+ = \{z : \mathrm{Re}\, z > 0\}$. Next $f(z) = h(z) + \overline{g(z)}$, where $h(z) = \log z + z$ and $g(z) = \log z - z$. Since $h'(z) = 1 + 1/z$ and $g'(z) = -1 + 1/z$, we have $|\nu(z)| < 1$ for $z \in \Pi^+$.

Moreover, $f$ is quasiconformal on every compact subset $D \subset \Pi^+$ and $\lambda_f$, $\Lambda_f$ are bounded from above and below on $D$. Therefore $f$ is a quasi-isometry on $D$ and by Theorem 2.19 below, $f$ satisfies the local uniform boundedness property on $D$.

From now on we consider the restriction of $f$ to $V = \{z = x + iy : x > 1, 0 < y < 1\}$. Then $fV = \{w = (u, v) : u > \log(1 + v^2/4), 0 < v < 2\}$.

We are going to show that:

- $f$ satisfies the local uniform boundedness property, but $f$ is not quasiconformal on $V$ .

We see that $f$ is not quasiconformal on $V$ , because $|\nu(z)| \to 1$ as $z \to \infty, z \in V$ .

For $s > 1$, define $V_s = \{z : 1 < x < s, 0 < y < 1\}$. Note that $f$ is qc on $V_s$ and therefore $f$ satisfies the property of local uniform boundedness on $V_s$ for every $s > 1$.

We consider separately two cases.

Case A. $z \in V_4$ . If $r > 1$ is big enough, then $d(z, \partial V_r) = d(z, \partial V)$ and $d\big(f(z), \partial f(V_r)\big) = d(f(z), \partial f(V))$ and therefore $f$ satisfies the property of local uniform boundedness on $V_4$ with respect to $r_V$.

Case B. It remains to prove that $f$ satisfies the property of local uniform boundedness on $V \setminus V_4$ with respect to $r_V$.

Observe first that for $z, z_1 \in V$ and $|z_1| \geq |z| \geq 1$ , we have the estimate

$$\log\left(\frac{|z_1|}{|z|}\right) \leq \frac{|z_1|}{|z|} - 1 \leq |z_1 - z|,$$

and therefore for $z, z_1 \in V$

$$|f(z_1) - f(z)| \leq 4|z_1 - z|. \tag{2.10}$$

We write

$$\partial V = [1, 1+i] \cup A \cup B\,;\, A = \{(x,0) : x \geq 1\}\,, B = \{(x,1) : x \geq 1\}\,.$$

Then

$$\partial(fV) = f(\partial V) \subset f[1,1+i] \cup (fA) \cup (fB)$$

and by the definition of $f$ we see that

$$fA = \{(x,0) : x \geq 0\}\,, \quad fB = \{(x,2) : x \geq \log 2\}, \quad f[1,1+i] \subset [0, \log 2] \times [0,2]\,.$$

Clearly for $w \in fV$

$$d(w) = \min\{d(w, fA), d(w, fB), d(w, f[1, 1+i])\}\,,$$

and for $\mathrm{Re}w > 1 + \log 2$ and $w \in fV$ , we find

$$d(w) = \min\{d(w, fA), d(w, fB)\}\,. \tag{2.11}$$

For $z \in V \setminus V_4$ we have $\mathrm{Re}f(z) \geq \log(16) > 1 + \log 2$ and therefore, in view of the definition of $f$, (2.11) yields $d(f(z)) = 2d(z)$. This together with (2.10) shows that $f$ satisfies the property of local uniform boundedness on $V \setminus V_4$. $\square$

## 2.4 Higher dimensional version of Schwarz lemma

Before giving a proof of the higher dimensional version of the Schwarz lemma for harmonic maps we first establish some notation.

Suppose that $h : \overline{B}^n(a,r) \to \mathbb{R}^n$ is a continuous vector-valued function, harmonic on $B^n(a,r)$, and let

$$M_a^* = \sup\{|h(y) - h(a)| : y \in S^{n-1}(a,r)\}.$$

Let $h = (h^1, h^2, \ldots, h^n)$. A modification of the estimate in [14, Equation (2.31)] gives

$$r|\nabla h^k(a)| \leq nM_a^*\,, \quad k = 1, \ldots, n.$$

We next extend this result to the case of vector valued functions. See also [8] and [5, Theorem 6.16].

**2.12. Lemma.** *Suppose that $h : \overline{B}^n(a,r) \to \mathbb{R}^n$ is a continuous mapping, harmonic in $B^n(a,r)$. Then*

$$r|h'(a)| \leq nM_a^*\,. \tag{2.13}$$

**Proof.** Without loss of generality, we may suppose that $a = 0$ and $h(0) = 0$. Let

$$K(x,y) = K_y(x) = \frac{r^2 - |x|^2}{n\omega_n\, r|x-y|^n},$$

where $\omega_n$ is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$.

Then

$$h(x) = \int_{S^{n-1}(0,r)} K(x,t)h(t)d\sigma, \quad x \in B^n(0,r),$$

where $d\sigma$ is the $(n-1)$-dimensional surface measure on $S^{n-1}(0,r)$.

A simple calculation yields

$$\frac{\partial}{\partial x_j}K(x,\xi) = \frac{1}{n\omega_n\, r}\left(\frac{-2x_j}{|x-\xi|^n} - n(r^2-|x|^2)\frac{x_j-\xi_j}{|x-\xi|^{n+2}}\right).$$

Hence, for $1 \le j \le n$, we have

$$\frac{\partial}{\partial x_j}K(0,\xi) = \frac{\xi_j}{\omega_n\, r^{n+1}}\,.$$

Let $\eta \in S^{n-1}$ be a unit vector and $|\xi| = r$. For given $\xi$, it is convenient to write $K_\xi(x) = K(x,\xi)$ and consider $K_\xi$ as a function of $x$.

Then

$$K'_\xi(0)\eta = \frac{1}{\omega_n\, r^{n+1}}(\xi,\eta)\,.$$

Since $|(\xi,\eta)| \le |\xi||\eta| = r$, we see that

$$|K'_\xi(0)\eta| \le \frac{1}{\omega_n\, r^n}\,, \text{ and therefore } \quad |\nabla K^\xi(0)| \le \frac{1}{\omega_n r^n}\,.$$

This last inequality yields

$$|h'(0)(\eta)| \le \int_{S^{n-1}(a,r)} |\nabla K^y(0)|\,|h(y)|\,d\sigma(y) \le \frac{M_0^*\, n\omega_n r^{n-1}}{\omega_n r^n} = \frac{M_0^* n}{r}$$

and the proof is complete. $\square$

Let $G \subset \mathbb{R}^n$, be a domain, let $h : G \to \mathbb{R}^n$ be continuous. For $x \in G$ let $B_x = B^n(x, \frac{1}{4}d(x))$ and

$$M_x = \omega_h(x) = \sup\{|h(y)-h(x)| : y \in B_x\}. \tag{2.14}$$

If $h$ is a harmonic mapping, then the inequality (2.13) yields

$$\frac{1}{4}d(x)|h'(x)| \le n\,\omega_h(x), \quad x \in G\,. \tag{2.15}$$

We also refer to (2.15) as the inner gradient estimate.

## 2.5 Harmonic quasiconformal quasi-isometries

For our purpose it is convenient to have the following lemma.

**2.16. Lemma.** *Let $G$ and $G'$ be two domains in $\mathbb{R}^n$, and let $\sigma$ and $\rho$ be two continuous metric densities on $G$ and $G'$, respectively, which define the elements of length $ds = \sigma(z)|dz|$ and $ds = \rho(w)|dw|$, respectively; and suppose that $f : G \to G'$, is a $C^1$-mapping.*
*a) If there is a positive constant $c_1$ such that $\rho(f(z))\,|f'(z)| \le c_1\,\sigma(z)$, $z \in G$, then $d_\rho(f(z_2), f(z_1)) \le c_1\, d_\sigma(z_2,z_1)$, $z_1, z_2 \in G$.*
*b) If $f(G) = G'$ and there is a positive constant $c_2$ such that $\rho(f(z))\,l(f'(z)) \ge c_2\,\sigma(z)$, $z \in G$, then $d_\rho(f(z_2), f(z_1)) \ge c_2\, d_\sigma(z_2,z_1)$, $z_1, z_2 \in G$*

The proof of this result is straightforward and it is left to the reader as an exercise.

**2.17. Pseudo-isometry and a quasi-isometry.** Let $f$ be a map from a metric space $(M, d_M)$ into another metric space $(N, d_N)$.

- We say that $f$ is a pseudo-isometry if there exist two positive constants $a$ and $b$ such that for all $x, y \in M$,

$$a^{-1}d_M(x,y) - b \le d_N(f(x), f(y)) \le a d_M(x,y).$$

- We say that $f$ is a quasi-isometry or a bi-Lipschitz mapping if there exists a positive constant $a \ge 1$ such that for all $x, y \in M$,

$$a^{-1}d_M(x,y) \le d_N(f(x), f(y)) \le a d_M(x,y).$$

For the convenience of the reader we begin our discusssion for the unit disk case.

**2.18. Theorem.** *Suppose that $h : \mathbb{D} \to \mathbb{R}^2$ is harmonic and satisfies the weak uniform boundedness property.*

(c) *Then $h : (\mathbb{D}, k_{\mathbb{D}}) \to (h(\mathbb{D}), k_{h(\mathbb{D})})$ is Lipschitz.*

(d) *If, in addition, $h$ is a qc mapping, then $h : (\mathbb{D}, k_{\mathbb{D}}) \to (h(\mathbb{D}), k_{h(\mathbb{D})})$ is a quasi-isometry.*

**Proof.** The part (d) is proved in [33].

For the proof of part (c) fix $x \in \mathbb{D}$ and $y \in B_x = B(x, \frac{1}{4}d(x))$. Then $d(y) \ge \frac{3}{4}d(x)$ and therefore $r(x,y) < 1/2$. By the hypotheses $|h(y) - h(x)| \le c\, d(h(x))$. The Schwarz lemma, applied to $B_x$ , yields in view of (2.14)

$$\frac{1}{4}d(x)|h'(x)| \le 2M_x \le 2c\, d(h(x))$$

The proof of part (c) follows from Lemma 2.16. □

A similar proof applies for higher dimensions; the following result is a generalization of the part $(c)$ of Theorem 2.18 .

**2.19. Theorem.** *Suppose that $G$ is a proper subdomain of $\mathbb{R}^n$ and $h : G \to \mathbb{R}^n$ is a harmonic mapping. Then the following conditions are equivalent*

(1) *$h$ satisfies the weak uniform boundedness property.*

(2) *$h : (G, k_G) \to (h(G), k_{h(G)})$ is Lipschitz.*

**Proof.** Let us prove that (1) implies (2).

By the hypothesis (1) $f$ satisfies the weak uniform boundedness property: for every $x \in G$ and $t \in B_x$

$$|f(t) - f(x)| \le c_2\, d(f(x))\,. \tag{2.20}$$

This inequality together with Lemma 2.12 gives $d(x)|f'(x)| \le c_3\, d(f(x))$ for every $x \in G$. Now an application of Lemma 2.16 shows that (1) implies (2).

It remains to prove that (2) implies (1).

Suppose that $f$ is Lipschitz with the multiplicative constant $c_2$. Fix $x, y \in G$ with $r_G(x,y) \le 1/2$. Then $|y-x| \le d(x)/2$ and therefore by Lemma 2.4

$$k_G(x,y) \le 2 j_G(x,y) \le 2\, r_G(x,y) \le 1.$$

Hence $k_{G'}(fx,fy) \le c_2$. Since $j_{G'}(fx,fy) \le k_{G'}(fx,fy) \le c_2$, we find $j_{G'}(fx,fy) = \log(1 + r_{G'}(fx,fy)) \le c_2$ and therefore $r_{G'}(fx,fy) \le e^{c_2} - 1$. $\square$

Since $f^{-1}$ is qc, an application of [12, Theorem 3] to $f^{-1}$ and Theorem 2.19 give the following corollary:

**2.21. Corollary.** *Suppose that $G$ is a proper subdomain of $\mathbb{R}^n$, $h : G \to hG$ is harmonic and $K$-qc. Then $h : (G, k_G) \to (h(G), k_{h(G)})$ is a pseudo-isometry.*

In [45, Example 11.4] (see also [44, Example 3.10]), it is shown that the analytic function $f : \mathbb{D} \to G, G = \mathbb{D} \setminus \{0\}, f(z) = \exp((z+1)/(z-1))$, $f(\mathbb{D}) = G$, fails to be uniformly continuous as a map

$$f : (\mathbb{D}, k_{\mathbb{D}}) \to (G, k_G)\,.$$

Therefore bounded analytic functions do not satisfy the weak uniform boundedness property in general. The situation will be different for instance if the boundary of the image domain is a continuum containing at least two points. Note that if $k_G$ is replaced by the hyperbolic metric $\lambda_G$ of $G$, then $f : (\mathbb{D}, k_{\mathbb{D}}) \to (G, \lambda_G)$ is Lipschitz.

**2.22. Theorem.** *Suppose that $G \subset \mathbb{R}^n$, $f : G \to \mathbb{R}^n$ is $K$-qr and $G' = f(G)$. Let $\partial G'$ be a continuum containing at least two distinct points. If $f$ is a harmonic mapping, then $f : (G, k_G) \to (G', k_{G'})$ is Lipschitz.*

**Proof.** Fix $x \in G$ and let $B_x = B^n(x, d(x)/4)$. If $|y-x| \le d(x)/4$, then $d(y) \ge 3d(x)/4$ and therefore,

$$r_G(y,x) \le \frac{4}{3}\frac{|y-x|}{d(x)}.$$

Because $j_G(x,y) = \log(1 + r_G(x,y)) \le r_G(x,y)$, using Lemma 2.4(a), we find

$$k_G(y,x) \le 2\, j_G(y,x) \le 2/3 < 1.$$

By [45, Theorem 12.21] there exists a constant $c_2 > 0$ depending only on $n$ and $K$ such that

$$k_{G'}(fy,fx) \le c_2 \max\{k_G(y,x)^{\alpha}, k_G(y,x)\}, \alpha = K^{1/(1-n)},$$

and hence, using Lemma 2.4(b) and $k_G(y,x) \le 1$, we see that

$$|fy - fx| \le e^{c_2} d(fx), \quad \text{i.e.} \quad M_x = \omega_f(x) \le e^{c_2}\, d(fx). \tag{2.23}$$

By (2.15) applied to $B_x = B^n(x, d(x)/4)$, we have

$$\frac{1}{4}d(x)|f'(x)| \leq 2M_x$$

and therefore using the inequality (2.23), we have

$$\frac{1}{4}d(x)|f'(x)| \leq 2\,c\,d(f(x)),$$

where $c = e^{c_2}$; and the proof follows from Lemma 2.16. $\square$

The first author has asked the following Question (cf. [33]: Suppose that $G \subset \mathbb{R}^n$ is a proper subdomain, $f : G \to \mathbb{R}^n$ is harmonic $K$-qc and $G' = f(G)$. Determine whether $f$ is a quasi-isometry w.r.t. quasihyperbolic metrics on $G$ and $G'$. This is true for $n = 2$ (see Theorem 2.26 below). It seems that one can modify the proof of Proposition 4.6 in [42] and show that this is true for the unit ball if $n \geq 3$ and $K < 2^{n-1}$, cf. also [20].

## 2.6 Quasi-isometry in planar case

Astala and Gehring [4] proved a quasiconformal analogue of Koebe's theorem, stated here as Theorem 2.24. These concern the quantity

$$a_f(x) = a_{f,G}(x) := \exp\left(\frac{1}{n|B_x|}\int_{B_x} \log J_f(z)dz\right), x \in G,$$

associated with a quasiconformal mapping $f : G \to f(G) \subset \mathbb{R}^n$; here $J_f$ is the Jacobian of f ; while $B_x$ stands for the ball $B(x; d(x, \partial G)$; and $|B_x|$ for its volume.

**2.24. Theorem[4].** *Suppose that $G$ and $G'$ are domains in $R^n$: If $f : G \to G'$ is $K$- quasiconformal, then*

$$\frac{1}{c}\frac{d(fx, \partial G')}{d(x, \partial G)} \leq a_{f,G}(x) \leq c\frac{d(fx, \partial G')}{d(x, \partial G)}, \quad x \in G,$$

*where $c$ is a constant which depends only on $K$ and $n$.*

Let $\Omega \in \mathbb{R}^n$ and $\mathbb{R}^+ = [0, \infty)$. If $f, g : \Omega \to \mathbb{R}^+$ and there is a positive constant $c$ such that

$$\frac{1}{c}\,g(x) \leq f(x) \leq c\,g(x)\,, \quad x \in \Omega\,,$$

we write $f \approx g$ on $\Omega$.

Our next result concerns the quantity

$$E_{f,G}(x) := \frac{1}{|B_x|}\int_{B_x} J_f(z)dz\,, x \in G,$$

associated with a quasiconformal mapping $f : G \to f(G) \subset \mathbb{R}^n$; here $J_f$ is the Jacobian of $f$ ; while $B_x$ stands for the ball $B(x, d(x, \partial G)/2$ and $|B_x|$ for its volume.

Define

$$A_{f,G} = \sqrt[n]{E_{f,G}}\,.$$

**2.25. Theorem.** *Suppose $f : \Omega \to \Omega'$ is a $C^1$ qc homeomorphism. The following conditions are equivalent:*

*a)$f$ is bi-Lipschitz with respect to quasihyperbolic metrics on $\Omega$ and $\Omega'$ ,*

*b) $\sqrt[n]{J_f} \approx d_*/d$ ,*

*c) $\sqrt[n]{J_f} \approx a_f$ ,*

*d) $\sqrt[n]{J_f} \approx A_f$,*

*where $d(x) = d(x, \partial\Omega)$ and $d_*(x) = d(f(x), \partial\Omega')$.*

**Proof.** It is known that a) is equivalent to b) (see for example [36]).

In [36], using Gehring's result on the distortion property of qc maps (see [10], p.383; [43], p.63), the first author gives short proofs of a new version of quasiconformal analogue of Koebe's theorem; it is proved that $A_f \approx d_*/d$.

By Theorem 2.24, $a_f \approx d_*/d$ and therefore b) is equivalent to c). The rest of the proof is straightforward. □

If $\Omega$ is planar domain and $f$ a harmonic qc map, then we proved that the condition d) holds.

The next theorem is a short proof of a recent result of V. Manojlovic [30],see also [33].

**2.26. Theorem.** *Suppose $D$ and $D'$ are proper domains in $\mathbb{R}^2$. If $h : D \to D'$ is $K$- qc and harmonic, then it is bi-Lipschitz with respect to quasihyperbolic metrics on $D$ and $D'$.*

**Proof.** Without loss of generality, we may suppose that $h$ is preserving orientation. Let $z \in D$ and $h = f + \overline{g}$ be a local representation of $h$ on $B_z$, where $f$ and $g$ are analytic functions on $B_z$, $\Lambda_h(z) = |f'(z)| + |g'(z)|$, $\lambda_h(z) = |f'(z)| - |g'(z)|$ and $k = \frac{K-1}{K+1}$.

Since $h$ is $K$- qc, we see that

$$(1-k^2)|f'|^2 \le J_h \le K|f'|^2 \tag{2.27}$$

on $B_z$ and since $\log|f'(\zeta)|$ is harmonic,

$$\log|f'(z)| = \frac{1}{2|B_z|}\int_{B_z} \log|f'(\zeta)|^2 d\xi\, d\eta\,.$$

Hence, using the right hand side of (2.27), we find

$$\log a_{h,D}(z) \le \frac{1}{2}\ \log K + \frac{1}{2|B_z|}\int_{B_z} \log|f'(\zeta)|^2 d\xi\, d\eta \tag{2.28}$$

$$= \ \log\sqrt{K}\,|f'(z)|\,. \tag{2.29}$$

Hence,

$$a_{h,D}(z) \le \sqrt{K}\,|f'(z)|$$

and in a similar way using the left hand side of (2.27), we have

$$\sqrt{1-k^2}\,|f'(z)| \le a_{h,D}(z)\,.$$

Now, an application of the Astala-Gehring result gives

$$\Lambda_h(z) \asymp \frac{d(hz, \partial D')}{d(z, \partial D} \asymp \lambda_h(z)\,.$$

This pointwise result, combined with Lemma 2.16 (integration along curves), easily gives

$$k_{D'}(h(z_1), h(z_2)) \asymp k_D(z_1, z_2), \quad z_1, z_2 \in D\,.$$

□

Note that in [30] the proof makes use of the interesting fact that $\log\frac{1}{J_h}$ is a subharmonic function; but we do not use it here.

Define $m_f(x,r) = \min\{|f(x') - f(x)| : |x' - x| = r\}$.

Suppose that $G$ and $G'$ are domains in $\mathbb{R}^n$: If $f : G \to G'$ is $K$- quasiconformal; by the distortion property we find $m_f(x,r) \ge a(x) r^{1/\alpha}$. Hence, as in [20] and [36] , we get:

**2.30. Lemma.** *If $f \in C^{1,1}$ is a $K-$ quasiconformal mapping defined in a domain $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$, then*

$$J_f(x) > 0, \quad x \in \Omega$$

*provided that $K < 2^{n-1}$. The constant $2^{n-1}$ is sharp.*

**2.31. Theorem.** *Under the hypothesis of the lemma, if $\overline{G} \subset \Omega$, then $f$ is bi-Lipschitz with respect to Euclidean and quasihyperbolic metrics on $G$ and $G' = f(G)$.*

**Proof.** Since $\overline{G}$ is compact $J_f$ attains minimum on $\overline{G}$ at a point $x_0 \in \overline{G}$. By Lemma 2.30, $m_0 = J_f > 0$ and therefore since $f \in C^{1,1}$ is a $K-$ quasiconformal, we conclude that functions $|f_{x_k}|$, $1 \le k \le n$ are bounded from above and below on $\overline{G}$; hence $f$ is bi-Lipschitz with respect to Euclidean metric on $G$.

By Theorem 2.24, we find $a_{f,G} \approx d_*/d$, where $d(x) = d(x, \partial G)$ and $d_*(x) = d(f(x), \partial G')$. Since we have here $\sqrt[n]{J_f} \approx a_f$, we find $\sqrt[n]{J_f} \approx d_*/d$ on $G$. An application of Theorem 2.25 completes the proof. □

## 2.7 The upper half space $\mathbb{H}^n$.

Let $\mathbb{H}^n$ denote the half-space in $\mathbb{R}^n$. If $D$ is a domain in $\mathbb{R}^n$, by $QCH(D)$ we denote the set of Euclidean harmonic quasiconformal mappings of $D$ onto itself.

In particular if $x \in \mathbb{R}^3$, we use notation $x = (x_1, x_2, x_3)$ and we denote by $\partial_{x_k} f = f'_{x_k}$ the partial derivative of $f$ with respect to $x_k$ .

A fundamental solution in space $\mathbb{R}^3$ of the Laplace equation is $\frac{1}{|x|}$. Let $U_0 = \frac{1}{|x+e_3|}$, where $e_3 = (0,0,1)$. Define $h(x) = (x_1 + \varepsilon_1 U_0, x_2 + \varepsilon_2 U_0, x_3)$. It is easy to verify that $h \in QCH(\mathbb{H}^3)$ for small values of $\varepsilon_1$ and $\varepsilon_2$.

Using the Herglotz representation of a nonnegative harmonic function $u$ (see Theorem 7.24 and Corollary 6.36 [5]), one can get:

**Lemma A**. If $u$ is a nonnegative harmonic function on a half space $\mathbb{H}^n$, continuous up to the boundary with $u = 0$ on $\mathbb{H}^n$, then $u$ is (affine) linear.

In [33], the first author has outlined a proof of the following result:

**Theorem A**. If $h$ is a quasiconformal harmonic mapping of the upper half space $\mathbb{H}^n$ onto itself and $h(\infty) = \infty$, then $h$ is quasi-isometry with respect to both the Euclidean and the Poincaré distance.

Note that the outline of proof in [33] can be justified by Lemma A.

We show that the analog statement of this result holds for $p$-harmonic vector functions (solutions of $p$-Laplacian equations) using the mentioned result obtained in the paper [23], stated here as:

**Theorem B**. If $u$ is a nonnegative $p$-harmonic function on a half space $\mathbb{H}^n$, continuous up to the boundary with $u = 0$ on $\mathbb{H}^n$, then $u$ is (affine) linear.

**2.32. Theorem.** *If $h$ is a quasiconformal $p$-harmonic mapping of the upper half space $\mathbb{H}^n$ onto itself and $h(\infty) = \infty$, then both $h : (\mathbb{H}^n, |\cdot|) \to (\mathbb{H}^n, |\cdot|)$ and $h : (\mathbb{H}^n, \rho_{\mathbb{H}^n}) \to (\mathbb{H}^n, \rho_{\mathbb{H}^n})$ are bi-Lipschitz where $\rho = \rho_{\mathbb{H}^n}$ is the Poincaré distance.*

Since 2-harmonic mapping are Euclidean harmonic this result includes Theorem A.

**Proof.** It suffices to deal with the case $n = 3$ as the proof for the general case is similar. Let $h = (h_1, h_2, h_3)$.

By Theorem B, we get $h_3(x) = ax_3$, where $a$ is a positive constant. Without loss of generality we may suppose that $a = 1$.

Since $h_3(x) = x_3$, we have $\partial_{x_3} h_3(x) = 1$, and therefore $|h'_{x_3}(x)| \geq 1$. In a similar way, $|g'_{x_3}(x)| \geq 1$, where $g = h^{-1}$. Hence, there exists a constant $c = c(K)$,

$$|h'(x)| \leq c \quad \text{and} \quad 1/c \leq l(h'(x))\,.$$

Therefore partial derivatives of $h$ and $h^{-1}$ are bounded from above; and, in particular, $h$ is Euclidean bi-Lipschitz.

Since $h_3(x) = x_3$,

$$\frac{|h'(x)|}{h_3(x)} \leq \frac{c}{x_3};$$

and hence, by Lemma 2.16, $\rho(h(a), h(b)) \leq c\rho(a, b)\,.$ $\square$

# 3 Pseudo-isometry and $OC^1(G)$

In this section, we give a sufficient condition for a qc mapping $f : G \to f(G)$ to be a pseudo-isometry w.r.t. quasihyperbolic metrics on $G$ and $f(G)$. First we adopt the following notation.

If $V$ is a subset of $\mathbb{R}^n$ and $u : V \to \mathbb{R}^m$, we define

$$\mathrm{osc}_V u = \sup\{|u(x) - u(y)| \,:\, x, y \in V\}\,.$$

Suppose that $G \subset \mathbb{R}^n$ and $B_x = B(x, d(x)/2)$. Let $OC^1(G)$ denote the class of $f \in C^1(G)$ such that

$$d(x)|f'(x)| \le c_1 \operatorname{osc}_{B_x} f \tag{3.1}$$

for every $x \in G$. Similarly, let $SC^1(G)$ be the class of functions $f \in C^1(G)$ such that

$$|f'(x)| \le a r^{-1} \omega_f(x, r) \quad \text{for all } B^n(x, r) \subset G, \tag{3.2}$$

where $\omega_f(x, r) = \sup\{|f(y) - f(x)| : y \in B^n(x, r)\}$.

The proof of Theorem 2.19 gives the following more general result:

**3.3. Theorem.** *Suppose that $G \subset \mathbb{R}^n$, $f : G \to G'$, $f \in OC^1(G)$ and it satisfies the weak property of uniform boundedness with a constant $c$ on $G$. Then*

(e) *$f : (G, k_G) \to (G', k_{G'})$ is Lipschitz.*

(f) *In addition, if $f$ is $K$-qc, then $f$ is pseudo-isometry w.r.t. quasihyperbolic metrics on $G$ and $f(G)$.*

**Proof.** By the hypothesis $f$ satisfies the weak property of uniform boundedness: $|f(t) - f(x)| \le c_2\, d(f(x)$ for every $t \in B_x$, that is

$$\operatorname{osc}_{B_x} f \le c_2\, d(f(x)) \tag{3.4}$$

for every $x \in G$. This inequality together with (3.1) gives $d(x)|f'(x)| \le c_3\, d(f(x))$. Now an application of Lemma 2.16 gives part (e). Since $f^{-1}$ is qc, an application of [12, Theorem 3] on $f^{-1}$ gives part (f). $\square$

In order to apply the above method we introduce subclasses of $OC^1(G)$ (see, for example, below (3.5)).

Let $f : G \to G'$ be a $C^2$ function and $B_x = B(x, d(x)/2)$. We denote by $OC^2(G)$ the class of functions which satisfy the following condition:

$$\sup_{B_x} d^2(x)|\Delta f(x)| \le c \operatorname{osc}_{B_x} f \tag{3.5}$$

for every $x \in G$.

If $f \in OC^2(G)$, then by Theorem 3.9 in [14], applied to $\Omega = B_x$,

$$\sup_{t \in B_x} d(t)|f'(t)| \le C(\sup_{t \in B_x} |f(t) - f(x)| + \sup_{t \in B_x} d^2(t)|\Delta f(t)|)$$

and hence by (3.5)

$$d(x)|f'(x)| \le c_1 \operatorname{osc}_{B_x} f \tag{3.6}$$

for every $x \in G$ and therefore $OC^2(G) \subset OC^1(G)$.

Now the following result follows from the previous theorem.

**3.7. Corollary.** *Suppose that $G \subset \mathbb{R}^n$ is a proper subdomain, $f : G \to G'$ is $K$-qc and $f$ satisfies the condition (3.5). Then $f : (G, k_G) \to (G', k_{G'})$ is Lipschitz.*

We will now give some examples of classes of functions to which Theorem 3.3 is applicable. Let $SC^2(G)$ denote the class of $f \in C^2(G)$ such that

$$|\Delta f(x)| \leq ar^{-1} \sup\{|f'(y)| : y \in B^n(x,r)\},$$

for all $B^n(x,r) \subset G$, where $a$ is a positive constant. Note that the class $SC^2(G)$ contains every function for which $d(x)|\Delta f(x)| \leq a|f'(x)|$, $x \in G$. It is clear that $SC^1(G) \subset OC^1(G)$ and by the mean value theorem, $OC^2(G) \subset SC^2(G)$. For example, in [39] it is proved that $SC^2(G) \subset SC^1(G)$ and that the class $SC^2(G)$ contains harmonic functions, eigenfunctions of the ordinary Laplacian if $G$ is bounded, eigenfunctions of the hyperbolic Laplacian if $G = \mathbb{B}^n$ and thus our results are applicable for instance to these classes.

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